Calculus II - Day 16

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Goals for today:

- Reverse the process for finding common denominators to rewrite an integral as a sum of simpler parts.
- Manage existential dread.
- Integrate "rational functions" of the form $\frac{P(x)}{Q(x)}$ where P and Q are polynomials.

Important Building Blocks:

$$\int \frac{a}{bx+d}\,dx$$

Let $u = bx + d$, so $du = b\,dx$:

$$= \frac{a}{b}\int \frac{1}{u}\,du = \frac{a}{b}\ln|u| + C$$

Substitute $u = bx + d$ back:

$$= \frac{a}{b}\ln|bx+d| + C$$

So,

$$\int \frac{a}{bx+d}\,dx = \frac{a}{b}\ln|bx+d| + C$$

$$\int \frac{1}{x^2+a^2}\,dx$$

Rewrite the integrand:

$$= \int \frac{1}{a^2\left(\left(\frac{x}{a}\right)^2+1\right)}\,dx = \frac{1}{a^2}\int \frac{1}{\left(\frac{x}{a}\right)^2+1}\,dx$$

Let $u = \frac{x}{a}$, so $du = \frac{1}{a}\,dx$, and thus $dx = a\,du$:

$$= \frac{1}{a^2}\int \frac{a}{u^2+1}\,du = \frac{1}{a}\int \frac{1}{u^2+1}\,du$$

Now integrate:

$$= \frac{1}{a} \arctan(u) + C$$

Substitute back $u = \frac{x}{a}$:

$$= \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

Thus,

$$\int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

Example:

$$\int \frac{x+5}{x^2 + x - 2}\, dx$$

We start by simplifying the integrand:

$$\frac{x+5}{x^2+x-2} = \frac{x+5}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

for some constants A and B.

This process is called **partial fraction decomposition (PFD)**.

Multiply both sides by the denominator:

$$x + 5 = [(x+2)(x-1)] \left[\frac{A}{x+2} + \frac{B}{x-1}\right] = A(x-1) + B(x+2)$$

Expanding the equation:

$$x + 5 = Ax - A + Bx + 2B = (A+B)x + (-A+2B)$$

Now, we can equate the coefficients:
- For the linear term: $1 = A + B$ - For the constant term: $5 = -A + 2B$
Solving these equations, we find:

$$A = -1, \quad B = 2$$

So,

$$\frac{x+5}{x^2+x-2} = \frac{-1}{x+2} + \frac{2}{x-1}$$

which means

$$\int \frac{x+5}{x^2+x-2}\, dx = \int \left(-\frac{1}{x+2} + \frac{2}{x-1}\right) dx$$

Now integrate each term:

$$= -\ln|x+2| + 2\ln|x-1| + C$$

Thus, the final answer is:

$$\int \frac{x+5}{x^2+x-2}\, dx = -\ln|x+2| + 2\ln|x-1| + C$$

Another Method:

Start with:

$$x + 5 = A(x-1) + B(x+2)$$

Set $x = -2$:

$$-2 + 5 = A(-3) + B(0) \Rightarrow 3 = -3A \Rightarrow A = -1$$

Set $x = 1$:

$$1 + 5 = A \cdot 0 + B \cdot 3 \Rightarrow 6 = 3B \Rightarrow B = 2$$

Thus, we have found:

$$A = -1, \quad B = 2$$

Procedure:

Let $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials with $\deg(P) < \deg(Q)$.

Assume:

$$Q(x) = a(x - r_1)(x - r_2) \dots (x - r_k)$$

where all roots are distinct and there are no irreducible quadratic factors.

- **Step 1:** Factor $Q(x)$ as

$$Q(x) = a(x - r_1)(x - r_2) \dots (x - r_k)$$

- **Step 2:** Rewrite $f(x)$ as

$$\frac{P(x)}{Q(x)} = \frac{A}{x - r_1} + \frac{B}{x - r_2} + \frac{C}{x - r_3} + \cdots + \frac{K}{x - r_k}$$

  where $A, B, C, \dots, K$ are constants to be determined.

- **Step 3:** Multiply both sides by $Q(x)$ to "clear denominators."
- **Step 4:** Solve the resulting system of equations to find the values of $A, B, C, \dots, K$.

Example:

$$\int \frac{9x^2 + 63x}{(x^2 - 5x + 4)(x + 2)}\, dx$$

1) Factor $Q(x)$:

$$Q(x) = (x - 4)(x - 1)(x + 2)$$

2) Write the form of the partial fraction decomposition (PFD):

$$\frac{9x^2+63x}{(x-4)(x-1)(x+2)} = \frac{A}{x-4} + \frac{B}{x-1} + \frac{C}{x+2}$$

3) Multiply through by $Q(x)$ to clear the denominators:

$$9x^2 + 63x = A(x-1)(x+2) + B(x-4)(x+2) + C(x-4)(x-1)$$

4) Solve by plugging in roots:
For $x = 4$:

$$9(4)^2 + 63(4) = A(4-1)(4+2) \Rightarrow 144 + 252 = A \cdot 3 \cdot 6$$

$$396 = 18A \Rightarrow A = \frac{396}{18} = 22$$

For $x = 1$:

$$9(1)^2 + 63(1) = B(1-4)(1+2) \Rightarrow 9 + 63 = B \cdot (-3) \cdot 3$$

$$72 = -9B \Rightarrow B = -8$$

For $x = -2$:

$$9(-2)^2 + 63(-2) = C(-2-4)(-2-1) \Rightarrow 36 - 126 = C \cdot (-6) \cdot (-3)$$

$$-90 = 18C \Rightarrow C = -5$$

So,

$$f(x) = \frac{22}{x-4} - \frac{8}{x-1} - \frac{5}{x+2}$$

Thus,

$$\int f(x)\,dx = 22\ln|x-4| - 8\ln|x-1| - 5\ln|x+2| + C$$

$$\boxed{\int \frac{9x^2+63x}{(x-4)(x-1)(x+2)}\,dx = 22\ln|x-4| - 8\ln|x-1| - 5\ln|x+2| + C}$$

What if we have repeated linear roots of $Q(x)$?

If $Q(x)$ has a root r that is repeated k times, so that $(x-r)^k$ appears as a factor of $Q(x)$, then instead of just having a term $\frac{A}{x-r}$, we need to include additional terms for each power up to k.

Specifically, the partial fraction decomposition would include terms of the form:

$$\frac{A}{x-r} + \frac{B}{(x-r)^2} + \frac{C}{(x-r)^3} + \cdots + \frac{K}{(x-r)^k}$$

where $A, B, C, \ldots, K$ are constants to be determined.

Example:

$$\int \frac{5x^2 - 3x + 2}{x^3 - 2x^2} \, dx$$

1) Factor $Q(x)$:

$$Q(x) = x^2(x-2)$$

2) Write the form of the partial fraction decomposition (PFD):

$$\frac{5x^2 - 3x + 2}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

3) Clear the denominators by multiplying both sides by $x^2(x-2)$:

$$5x^2 - 3x + 2 = A \cdot x(x-2) + B(x-2) + Cx^2$$

4) Plug in the roots $x = 0$ and $x = 2$ to solve for B and C:
For $x = 0$:

$$5(0)^2 - 3(0) + 2 = B(0-2) \Rightarrow 2 = -2B \Rightarrow B = -1$$

For $x = 2$:

$$5(2)^2 - 3(2) + 2 = C \cdot (2)^2 \Rightarrow 16 = 4C \Rightarrow C = 4$$

5) To solve for A, we "compare like terms" on both sides of the equation. Expand the right-hand side and equate the coefficients of x^2:
Given $5 = A + C$ and $C = 4$:

$$5 = A + 4 \Rightarrow A = 1$$

Thus, the partial fraction decomposition is:

$$\frac{5x^2 - 3x + 2}{x^2(x-2)} = \frac{1}{x} - \frac{1}{x^2} + \frac{4}{x-2}$$

Now integrate term by term:

$$\int \frac{5x^2 - 3x + 2}{x^2(x-2)} \, dx = \int \left(\frac{1}{x} - \frac{1}{x^2} + \frac{4}{x-2} \right) dx$$

$$= \int \frac{1}{x} \, dx - \int \frac{1}{x^2} \, dx + \int \frac{4}{x-2} \, dx$$

$$= \ln|x| + \frac{1}{x} + 4\ln|x-2| + C$$

Final answer:

$$\boxed{\int \frac{5x^2 - 3x + 2}{x^2(x-2)} \, dx = \ln|x| + \frac{1}{x} + 4\ln|x-2| + C}$$

The corresponding term in a PFD for an irreducible quadratic $ax^2 + bx + c$ where $b^2 - 4ac < 0$ is

$$\frac{Ax + B}{ax^2 + bx + c}$$

Write down the form of the PFD for

$$f(x) = \frac{x^2+x-1}{(x-1)^2(x+3)^3x} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3} + \frac{D}{(x+3)^2} + \frac{E}{(x+3)^3} + \frac{F}{x}$$

Q: What if there's an irreducible quadratic?

Ex. $x^2 - 2x + 3$ can't be factored over $\mathbb{R}$

roots: $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{2\pm\sqrt{4-12}}{2}$ (negative... complex roots)

Example:

Consider the integral

$$\int \frac{x}{(x-2)(x^2+1)(x^2+4)^2}\,dx$$

Since the denominator includes an irreducible quadratic factor x^2+1 (where $b^2-4ac = -4 < 0$) and a repeated irreducible quadratic factor $(x^2+4)^2$, the form of the partial fraction decomposition is:

$$\frac{x}{(x-2)(x^2+1)(x^2+4)^2} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$$

To find the constants A, B, C, D, E, F, and G, multiply both sides by the denominator $(x-2)(x^2+1)(x^2+4)^2$ to clear the fractions:

$$x = A(x^2+1)(x^2+4)^2 + (Bx+C)(x-2)(x^2+4)^2 + (Dx+E)(x-2)(x^2+1)(x^2+4) + (Fx+G)(x-2)(x^2+1)$$

$$\int \left(\frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4} \right) dx$$

Rewrite the integral by splitting each term:

$$= \int \frac{A}{x-2}\,dx + \int \frac{Bx}{x^2+1}\,dx + \int \frac{C}{x^2+1}\,dx + \int \frac{Dx}{x^2+4}\,dx + \int \frac{E}{x^2+4}\,dx$$

$$= A\ln|x-2| + \frac{B}{2}\ln|x^2+1| + C\arctan(x) + \frac{D}{2}\ln|x^2+4| + \frac{E}{2}\arctan\left(\frac{x}{2}\right) + F$$

So, the final answer is:

$$\boxed{\int \left(\frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4} \right) dx = A\ln|x-2| + \frac{B}{2}\ln|x^2+1| + C\arctan(x) + \frac{D}{2}\ln|x^2+4| + \frac{E}{2}\arctan\left(\frac{x}{2}\right) + F}$$